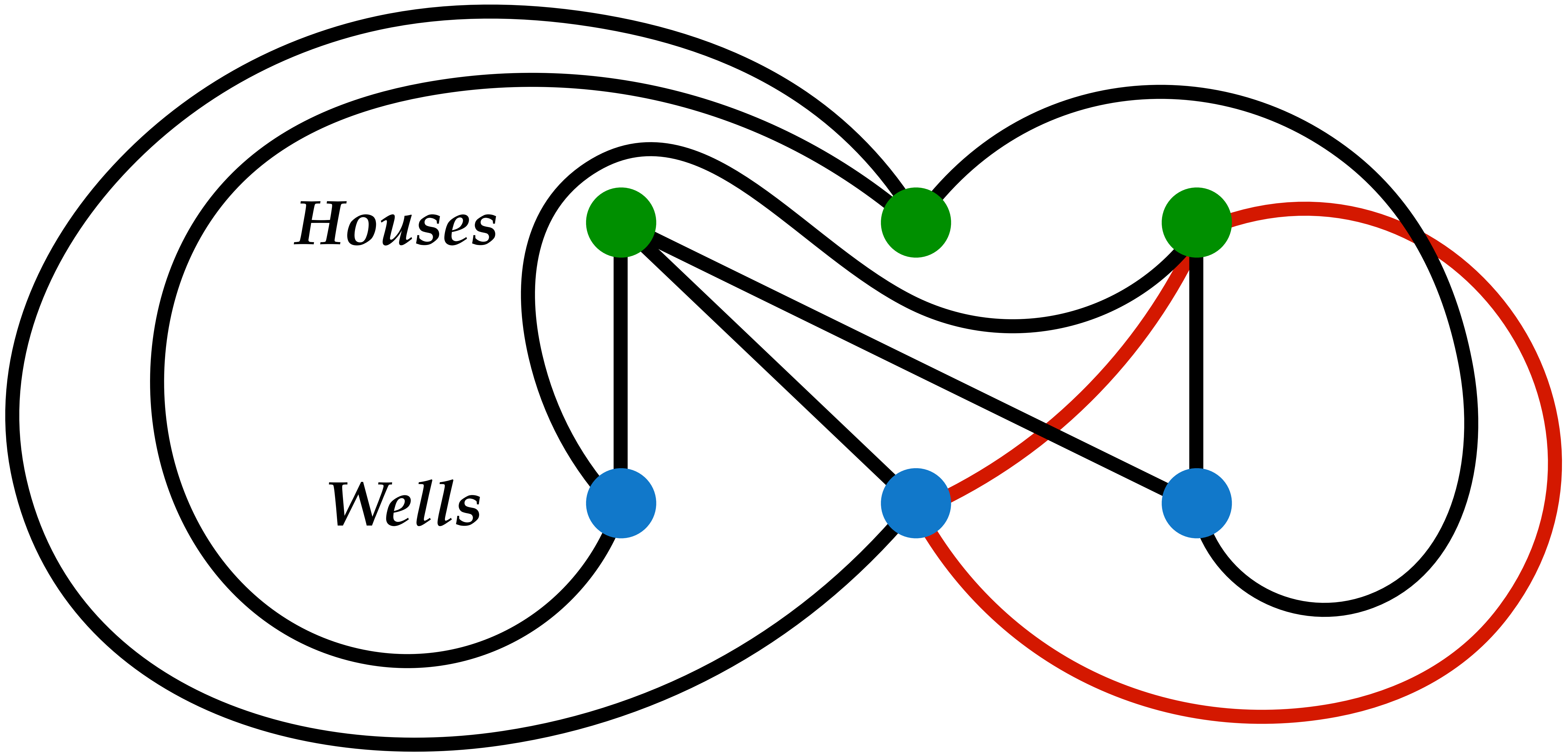
Lecture 34

Planar Graphs

Three Houses & Three Well

Suppose a farming community has three houses and three wells. The families living in there houses cannot stand each other, so they prefer to not meet when they walk to any well.

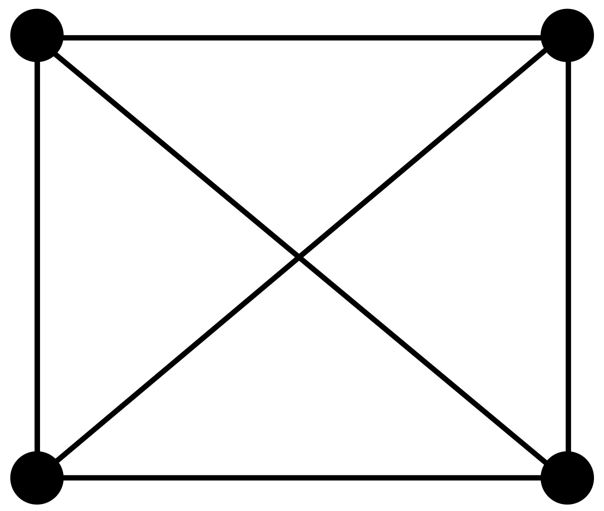
Can we build houses, wells, and roads in a way so that the families do not have to meet each other while walking to wells? No, we will prove that soon.



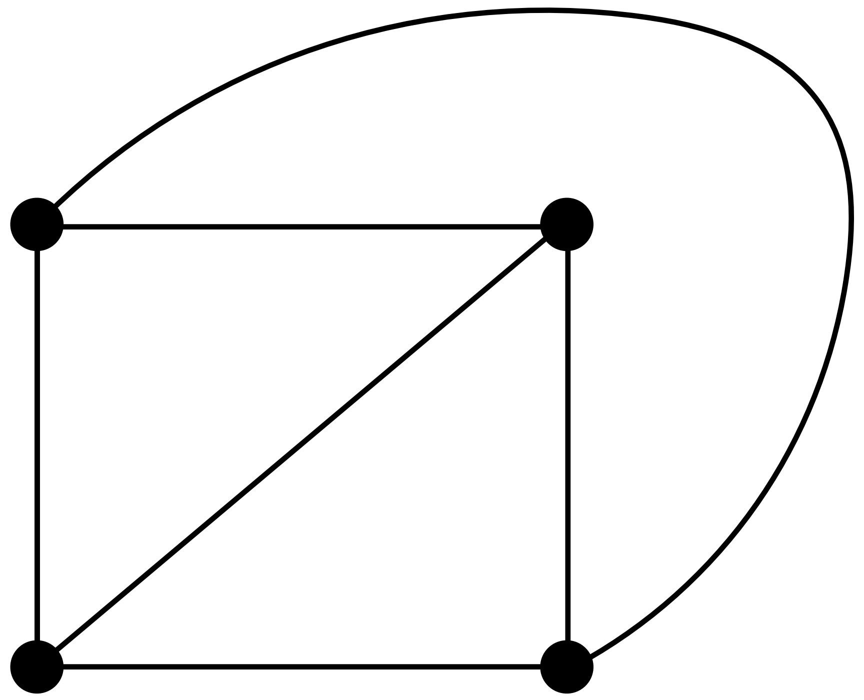
Planar Graphs

Definition: A graph is a **planar graph** if it can be drawn on the plane in such a way that its edges intersect with each other only at their endpoints. Such a drawing is called a **planar embedding** of the graph.

Example:



K_4



Planar embedding of K_4

Faces and Boundary Planar Graphs

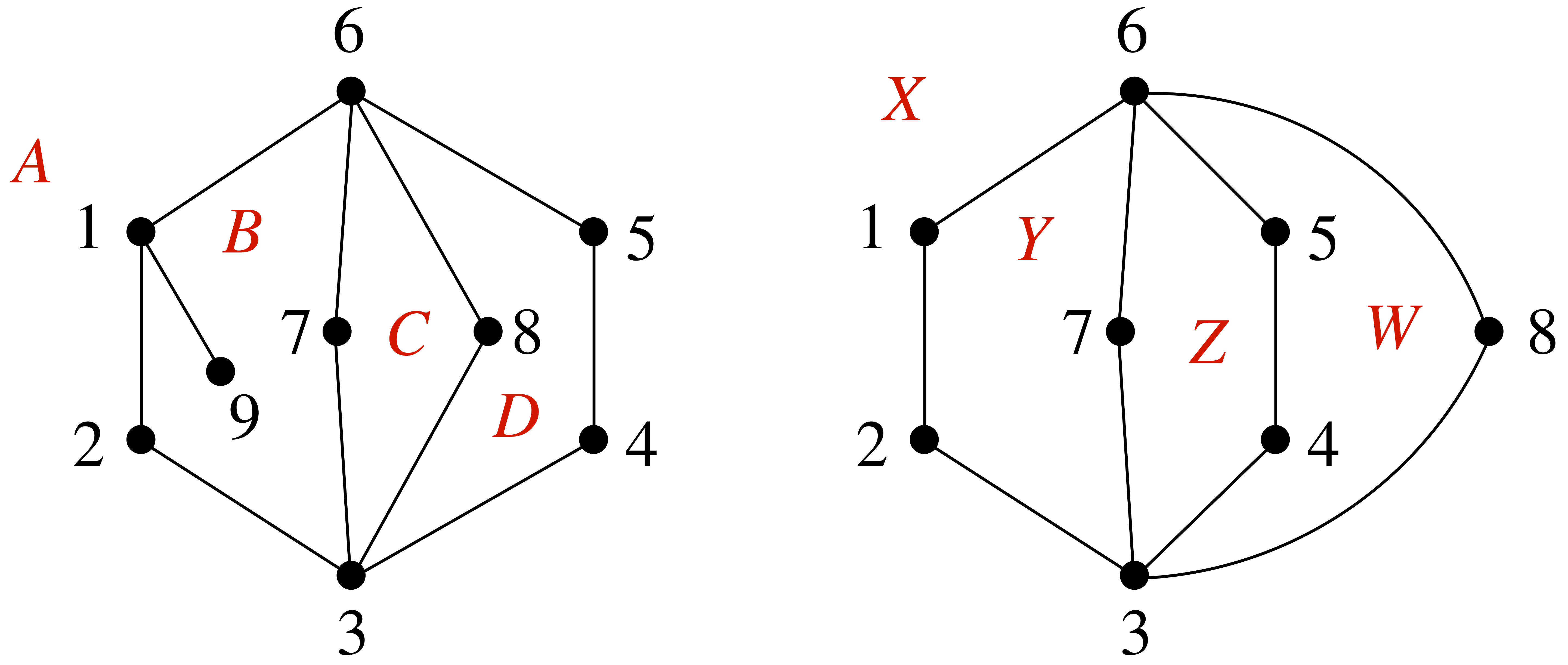
Definition: Let G be a planar graph. Then the edges of G in a planar embedding of G, partition the plane into regions. These regions are called as **faces** of G.

Every face is bounded by a sequence of edges called the **boundary** of that face.

Example:

$$A = \langle 1,2,3,4,5,6,1 \rangle$$

$$B = \langle 1,6,7,4,3,2,1,9,1 \rangle$$



Observation: Every edge lies either on the boundary of two faces or appears twice on the boundary of a face.

Euler's Theorem on Planar Graphs

Theorem: Let G be a connected planar graph with $|V|$ many vertices, $|E|$ many edges, and $|F|$ many faces (w.r.t. any embedding). Then, $|V| - |E| + |F| = 2$.

Proof: We will do induction on the number of edges.

Basis Step: For $|E| = 1$, the only connected graph is a tree of 2 vertices.

Hence, $|V| - |E| + |F| = 2 - 1 + 1 = 2$

Inductive Step: Assume the statement is true for all the graphs of $|E| - 1$ edges.

Let's take a graph G of $|V|$ vertices, $|E|$ edges, $|F|$ face (w.r.t. embedding $em(G)$).

Case 1: If there is a cycle in G, consider $G - e$, where e is an edge in that cycle.

e must be on the boundary of 2 different faces in $em(G)$.

Thus, $em(G) - e$ must have one face lesser than $em(G)$.

From IH, $|V| - (|E| - 1) + (|F| - 1) = 2 \implies |V| - |E| + |F| = 2$

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Euler's Theorem on Planar Graphs

Theorem: Let G be a connected planar graph with $|V|$ many vertices, $|E|$ many edges, and $|F|$ many faces (w.r.t. some embedding). Then, $|V| - |E| + |F| = 2$.

Proof: *Case* 2: If there are no cycles, then G is a tree.

In a tree, $|E| = |V| - 1$ and $|F| = 1$. Hence, the statement is true. ■